Discussion

Alexander Kruger

University of Ballarat, Australia

The article presents a report on a wonderful tour in the area of *stability analysis* of linear (and not only linear) optimization undertaken in the last 15 years by the author and his team of collaborators. 15 years is a very short period for developing a mathematical theory. Nevertheless the scope of achievement presented in the article and the level of development of the theory are really impressive. The tour is full of attractions and the route is very carefully marked. Now the tour is on offer, and the author is eager to share its highlights with interested travelers.

Linear optimization is the most practical and the most developed brunch of optimization theory. For people not directly involved in this area, it is difficult to comprehend that there is still room for new theories. Surprisingly the author and his collaborators have discovered quite a large niche full of important practical problems: so called *semi-infinite* and *infinite* linear programming problems, that is, linear optimization problems with infinite number of constraints and infinite number of variables (including, in particular, linear optimal control problems) respectively. Stability analysis of such problems is the main topic of the article.

For a new theory to be generally accepted, among other things it is important to develop convenient classification, terminology and notation which should agree with what is in use in neighboring areas. I believe that the stability theory of semi-infinite and infinite linear optimization presented by Marco López meets this criterion well. Furthermore, a number of concepts suggested in this and the preceding articles are applicable (and already in use) in other branches of optimization theory.

Another important feature of the presented theory which can surprise some readers is the extensive use of general *variational analysis* constructions. It might look unusual that these powerful tools designed for dealing with general nonlinear and even set-valued mappings are used when analyzing linear problems. However, this is very well justified and this article by Marco López provides a series of good examples demonstrating wide applicability of variational analysis techniques in linear optimization. It should be emphasized that despite all functions involved in the setting of a problem being linear, its *optimal value function* is in most cases nonlinear and nonsmooth while the *feasible* and *optimal set* mappings are obviously set-valued. Investigating *semicontinuity*, *Aubin Lipschitz-like property* and *regularity* of such mappings is important for stability analysis of linear and other optimization problems.

Studying *quantitative stability* of semi-infinite linear programming problems makes a significant part of the article. This reflects the general trend in optimization theory stimulated by the importance of this type of stability analysis providing numerical estimates of the corresponding properties. This article focuses mostly on characterizing the central variational analysis properties of *metric regularity/pseudo Lipschitzness* and related to them distance to *ill-posedness*. Another important property of *calmness* has been paid significantly less attention. It is introduced by inequalities (1.3) and (1.4) under the name of local and global *error bounds* respectively. A characterization of the latter property is given in Theorem 10. In terms of terminology, inequalities (1.3) and (1.4) definitely can be interpreted as error bounds and in the case of a real-valued function they are usually referred to as error bounds. In the general set-valued setting, this property is known in variational analysis as *calmness*. This term should have been mentioned in the article to avoid possible confusion.

It is quite natural to characterize regularity and Lipschitz-like properties in terms of certain constants, known as exact bounds, moduli, norms, rates, condition numbers, etc. (Rockafellar and Wets, 1998; Mordukhovich, 2006; Ioffe, 2000). In most (possibly all) cases such constants can be represented as limiting derivative-like objects (limits of "difference quotients"). In my opinion, this better reflects the nature of these properties and simplifies their comparison and characterization – see the discussion in (Dmitruk and Kruger, 2008; Kruger, 2009). I want to note as another positive thing that the presentation in the article by Marco López complies with this idea. For instance, the definition of the exact Lipschitz bound

$$\operatorname{lip} \mathcal{F}(\bar{\pi}, \bar{x}) = \limsup_{(x,\pi)\to(\bar{\pi},\bar{x})} \frac{d(x, \mathcal{F}(\pi))}{\mathbf{d}(\pi, \mathcal{F}^{-1}(x))}$$

in Subsection 1.4.2 coincides with the corresponding definition in (Dmitruk and Kruger, 2008).

I am strongly impressed by the breadth of research presented in the article. It seems no important questions remain unanswered. The theorems cover step by step the area announced in the title. Many theorems present long lists of equivalences fully characterizing corresponding properties. Fortunately the techniques developed for stability analysis in linear optimization are applicable to more general problems. After several successful outings into convex optimization, Marco López and his collaborators seems to be ready for crossing the borders of linear optimization on a massive scale. This is demonstrated in Section 3 of the article.

One of the factors contributing to the success of this project is the strong team of researchers who have succeeded in making Spain known to the world as one of the strong centres in the area of optimization and operations research. An important new development that has been happening in recent years is the process of expanding the team by including into it researchers from other countries (and continents) making it a very strong really international team.

References

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